SOME NEW IDENTITIES OF GENOCCHI NUMBERS AND POLYNOMIALS INVOLVING BERNOULLI AND EULER POLYNOMIALS

SERKAN ARACI, MEHMET ACIKGOZ, AND ERDOĞAN ŞEN

Abstract. In this paper, we will deal with some new formulae for product of two Genocchi polynomials together with both Euler polynomials and Bernoulli polynomials. We get some applications for Genocchi polynomials. Our applications possess a number of interesting properties to study in Theory of Analytic numbers which we express in the present paper.

2010 Mathematics Subject Classification. 11S80, 11B68.

Keywords and phrases. Genocchi numbers and polynomials, Bernoulli numbers and polynomials, Euler numbers and polynomials, Application.

1. Introduction

As is well known, the Genocchi polynomials are defined by the exponential generating function, as follows:

$$e^{G(x)t}=\sum_{n=0}^{\infty}G_n\left(x\right)\frac{t^n}{n!}=\frac{2t}{e^t+1}e^{xt} \tag{1}$$

with the usual convention about replacing $G^n\left(x\right):=G_n\left(x\right)$, symbolically. Taking $x=0$ into (1), then we have $G_n\left(0\right):=G_n$ which is called Genocchi numbers (for details, see [1], [2], [11], [13], [17], [19], [20], [24]). Differentiating both sides of (1), with respect to x, then we have the following:

$$\frac{d}{dx}G_n\left(x\right)=nG_{n-1}\left(x\right). \tag{2}$$

By (1) and (2), we can easily derive the following:

$$\int_b^a G_n\left(x\right)dx=\frac{G_{n+1}\left(a\right)-G_{n+1}\left(b\right)}{n+1}. \tag{3}$$

By (1), we get

$$G_n\left(x\right)=\sum_{k=0}^{n}\binom{n}{k}G_kx^{n-k}. \tag{4}$$

By (3) and (4), we can derive

$$\int_0^1 G_n\left(x\right)dx=-2\frac{G_{n+1}}{n+1}. \tag{5}$$

It is not difficult to see the following:

$$e^{tx} = \frac{1}{2t}\left(\frac{2t}{e^t+1}e^{(1+x)t}+\frac{2t}{e^t+1}e^{xt}\right) = \frac{1}{2t}\sum_{n=0}\left(G_n(x+1)+G_n(x)\right)\frac{t^n}{n!}. \tag{6}$$

By expression of (6), then we have

$$2nx^{n-1} = G_n(x+1)+G_n(x) \tag{7}$$

(see [11], [2]). By (7), we see that $\{G_0(x), G_1(x), \cdots, G_n(x)\}$ is the basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$.

In [8], Kim *et al.* introduced the following integrals:

$$I_{m,n} = \int_0^1 B_m(x)\, x^n dx \text{ and } J_{m,n} = \int_0^1 E_m(x)\, x^n dx \tag{8}$$

where $B_m(x)$ and $E_n(x)$ are called Bernoulli polynomials and Euler polynomials, respectively. Also, they are defined by the following generating functions:

$$e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt},\ \ |t|<2\pi, \tag{9}$$

$$e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt},\ \ |t|<\pi \tag{10}$$

with $B^n(x) := B_n(x)$ and $E^n(x) := E_n(x)$, symbolically. By substituting $x=0$ in (9) and (10), then we readily see that,

$$\sum_{n=0}^{\infty} B_n(0)\frac{t^n}{n!} = \frac{t}{e^t-1}, \tag{11}$$

$$\sum_{n=0}^{\infty} E_n(0)\frac{t^n}{n!} = \frac{2}{e^t+1}. \tag{12}$$

Here $B_n(0) := B_n$ and $E_n(0) := E_n$ are called Bernoulli numbers and Euler numbers, respectively. Thus, Bernoulli and Euler numbers and polynomials have the following identities:

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k} \text{ and } E_n(x) = \sum_{k=0}^{n}\binom{n}{k}E_k x^{n-k}. \tag{13}$$

By (11) and (12), we have the following recurrence relations of Euler and Bernoulli numbers, as follows:

$$B_0 = 1,\ B_n(1) - B_n = \delta_{1,n} \text{ and } E_0 = 1,\ E_n(1) + E_n = 2\delta_{0,n} \tag{14}$$

where $\delta_{n,m}$ is the Kronecker's symbol which is defined by

$$\delta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m. \end{cases} \tag{15}$$

From of Eqs (8-15), Kim *et al.* derived some new formulae on the product for two and several Bernoulli and Euler polynomials (for details, see [4-10]).

In [3], He and Wang also gave formulae of products of the Apostol-Bernoulli and Apostol-Euler Polynomials. With the help of their effectiveness works, we are motivated to write this paper. Thus, we also introduce some new interesting identities for Genocchi numbers and polynomials in the next section.

2. **On the Genocchi numbers and polynomials**

In this section, we introduce the following integral equation: For $m, n \geq 1$,

$$T_{m,n} = \int_0^1 G_m(x)\, x^n dx. \tag{16}$$

By (16), becomes:

$$T_{m,n} = -\frac{G_{m+1}}{m+1} - \frac{n}{m+1} \int_0^1 G_{m+1}(x)\, x^{n-1} dx.$$

Thus, we have the following recurrence formulas, as follows:

$$T_{m,n} = -\frac{G_{m+1}}{m+1} - \frac{n}{m+1} T_{m+1,n-1}$$

by continuing with the above recurrence relation, then we derive that

$$T_{m,n} = -\frac{G_{m+1}}{m+1} + (-1)^2 \frac{n}{(m+1)(m+2)} G_{m+2} + (-1)^2 \frac{n(n-1)}{(m+1)(m+2)} T_{m+2,n-2}.$$

Now also, we develop the following for sequel of this paper:

$$T_{m,n} = \frac{1}{n+1} \sum_{j=1}^{n} (-1)^j \frac{\binom{n+1}{j}}{\binom{m+j}{m}} G_{m+j} + 2 \frac{(-1)^{n+1} G_{n+m+1}}{(n+m+1)\binom{n+m}{m}}. \tag{17}$$

Let us now introduce the polynomial

$$p(x) = \sum_{l=0}^{n} G_l(x)\, x^{n-l}, \text{ with } n \in \mathbb{N}.$$

Taking k-th derivative of the above equality, then we have

$$\begin{aligned} p^{(k)}(x) &= (n+1)\, n\, (n-1) \cdots (n-k+2) \sum_{l=k}^{n} G_{l-k}(x)\, x^{n-l} \\ &= \frac{(n+1)!}{(n-k+1)!} \sum_{l=k}^{n} G_{l-k}(x)\, x^{n-l} \quad (k = 0, 1, 2, \cdots, n). \end{aligned} \tag{18}$$

On account of the properties of the Genocchi basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, then $p(x)$ can be written as follows:

$$p(x) = \sum_{k=0}^{n} a_k B_k(x) = a_0 + \sum_{k=1}^{n} a_k B_k(x). \tag{19}$$

Therefore, by (19), we obtain

$$\begin{aligned}
a_0 &= \int_0^1 p(x)\,dx = \sum_{k=1}^{n}\int_0^1 G_k(x)\,x^{n-k}dx = \sum_{k=1}^{n} T_{k,n-k} = \sum_{k=1}^{n-1} T_{k,n-k} + T_{k,0} \\
&= \sum_{k=1}^{n-1}\frac{1}{n-k+1}\sum_{j=1}^{n-k}(-1)^j\frac{\binom{n-k+1}{j}}{\binom{k+j}{k}}G_{k+j} + 2\frac{(-1)^{n-k+1}G_{n+1}}{(n+1)\binom{n}{k}} - 2\frac{G_{k+1}}{k+1}.
\end{aligned}$$

From expression of (18), we get

$$\begin{aligned}
a_k &= \frac{1}{k!}\left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) \\
&= \frac{(n+1)!}{k!\,(n-k+2)!}\left(\sum_{l=k-1}^{n} G_{l-k+1}(1) - 0^{n-l}G_{n-k+1}\right) \\
&= \frac{\binom{n+2}{k}}{n+2}\sum_{l=k-1}^{n-1}\left(2 - G_{l-k+1} - G_{n-k+1}\right).
\end{aligned}$$

From the above applications, we state the following theorem:

Theorem 2.1. *The following equality holds true:*

$$\begin{aligned}
&\sum_{l=0}^{n} G_l(x)\,x^{n-l} \\
&= \sum_{k=1}^{n-1}\left(\sum_{j=1}^{n-k}(-1)^j\frac{\binom{n-k+1}{j}}{(n-k+1)\binom{k+j}{k}}G_{k+j} + 2\frac{(-1)^{n-k+1}G_{n+1}}{(n+1)\binom{n}{k}} - 2\frac{G_{k+1}}{k+1}\right) \\
&\quad + \sum_{k=1}^{n}\left(\frac{\binom{n+2}{k}}{n+2}\sum_{l=k-1}^{n-1}\left(2 - G_{l-k+1} - G_{n-k+1}\right)\right)B_k(x).
\end{aligned}$$

Let us now consider the polynomial $p(x)$ as

$$p(x) = \sum_{k=0}^{n} b_k E_k(x).$$

In [8], Kim *et al.* gave the coefficients b_k by utilizing from the definition of Bernoulli polynomials. Now also, we give the coefficients b_k by using the definition of Genocchi polynomials, as follows:

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left(p^{(k)}(1) + p^{(k)}(0)\right) \\
&= \frac{(n+1)!}{2k!\,(n-k+1)!}\sum_{l=k}^{n}\left(G_{l-k}(1) + 0^{n-l}G_{l-k}\right) \\
&= (n+1)\binom{n}{k} - \frac{\binom{n+1}{k}}{2}\sum_{l=k}^{n-1}\left(G_{l-k} - G_{n-k}\right).
\end{aligned}$$

After these applications, then we can easily discover the following theorem:

Theorem 2.2. *The following nice identity*

$$\sum_{l=0}^{n} G_l(x)\, x^{n-l}$$
$$= \sum_{k=0}^{n} \left((n+1)\binom{n}{k} - \frac{\binom{n+1}{k}}{2} \sum_{l=k}^{n-1} \left(G_{l-k} - G_{n-k}\right) \right) E_k(x)$$

is true.

We now consider the following polynomial:

$$p(x) = \sum_{l=0}^{n} \frac{1}{l!\,(n-l)!} G_l(x)\, x^{n-l} = \sum_{l=0}^{n} a_l G_l(x).$$

It is not difficult to indicate the following:

$$p^{(k)}(x) = 2^k \sum_{l=k}^{n} \frac{1}{(l-k)!\,(n-l)!} G_{l-k}(x)\, x^{n-l}. \tag{20}$$

Then, we see that for $k = 1, 2, \cdots, n$,

$$\begin{aligned}
a_l &= \frac{1}{2l!}\left(p^{(l-1)}(1) + p^{(l-1)}(0)\right) \\
&= \frac{2^{l-2}}{l!} \sum_{j=l-1}^{n} \frac{1}{(j-l+1)!\,(n-j)!} \left(G_{j-l+1}(1) + 0^{n-j} G_{j-l+1}\right) \\
&= \frac{2^{l-2}}{l!} \sum_{j=l-1}^{n} \frac{(2 - G_{l-j+1})}{(j-l+1)!\,(n-j)!} + \frac{2^{l-2}}{l!\,(n-l+1)!} G_{n-l+1}.
\end{aligned}$$

So, we discover the following interesting and worthwhile theorem for studying in Analytic Numbers Theory.

Theorem 2.3. *The following equality holds:*

$$\sum_{l=0}^{n} \frac{1}{l!\,(n-l)!} G_l(x)\, x^{n-l}$$
$$= \sum_{l=1}^{n} \frac{2^{l-2}}{l!} \sum_{j=l-1}^{n} \frac{(2 - G_{l-j+1})\, G_l(x)}{(j-l+1)!\,(n-j)!} + \frac{2^{l-2}}{l!\,(n-l+1)!} G_{n-l+1} G_l(x).$$

Now also, let us take the polynomial in terms of Bernoulli polynomials as

$$p(x) = \sum_{k=0}^{n} a_k B_k(x).$$

By using the above identity, we develop as follows:

$$\begin{aligned} a_0 &= \int_0^1 p(x)\,dx = \sum_{l=0}^{n} \frac{1}{l!\,(n-l)!} \int_0^1 G_l(x)\, x^{n-l} dx \\ &= \sum_{l=0}^{n} \frac{1}{l!\,(n-l)!} T_{l,n-l} = T_{n,0} + \sum_{l=1}^{n-1} \frac{1}{l!\,(n-l)!} T_{l,n-l} \\ &= -2\frac{G_{n+1}}{n+1} + \sum_{l=1}^{n-1}\sum_{j=1}^{n-l} \frac{(-1)^j}{l!\,(n-l+1)!} \frac{\binom{n-l+1}{j}}{\binom{l+j}{l}} G_{l+j} + 2\frac{(-1)^{n-l+1} G_{n+1}}{(n+1)\binom{n}{l}}. \end{aligned}$$

By (20), we compute a_k coefficients, as follows:

$$\begin{aligned} a_k &= \frac{1}{k!}\left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) \\ &= \frac{2^{k-1}}{k!} \sum_{l=k-1}^{n} \frac{1}{(l-k+1)!\,(n-l)!} \left(G_{l-k+1}(1) - 0^{n-l} G_{l-k+1}\right) \\ &= \frac{2^{k-1}}{k!} \sum_{l=k-1}^{n} \frac{(2 - G_{l-k+1})}{(l-k+1)!\,(n-l)!} - \frac{2^{k-1}}{k!\,(n-k+1)!} G_{n-k+1}. \end{aligned}$$

Consequently, we state the following theorem.

Theorem 2.4. *The following identity*

$$\sum_{l=0}^{n} \frac{1}{l!\,(n-l)!} G_l(x)\, x^{n-l} \tag{21}$$

$$= -2\frac{G_{n+1}}{n+1} + \sum_{l=1}^{n-1}\sum_{j=1}^{n-l} \frac{(-1)^j}{l!\,(n-l+1)!} \frac{\binom{n-l+1}{j}}{\binom{l+j}{l}} G_{l+j} + 2\frac{(-1)^{n-l+1} G_{n+1}}{(n+1)\binom{n}{l}}$$

$$+\sum_{k=1}^{n}\left(\frac{2^{k-1}}{k!} \sum_{l=k-1}^{n} \frac{(2 - G_{l-k+1})}{(l-k+1)!\,(n-l)!} - \frac{2^{k-1}}{k!\,(n-k+1)!} G_{n-k+1}\right) B_k(x)$$

is true.

In [11], it is well-known that

$$G_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} G_k(x)\, y^{n-k}. \tag{22}$$

For $x = y$ in (22), then we have the following

$$\frac{1}{n!} G_n(2x) = \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} G_k(x)\, x^{n-k}. \tag{23}$$

By comparing the equations of (21) and (23), then we readily derive the following corollary.

Corollary 2.5.

$$\frac{1}{n!} G_n(2x) = \textit{the right-hand-side of equation in Theorem 2.4.}$$

Let us now introduce

$$p(x)=\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)\,x^{n-k}.$$

Then, we derive k-th derivative of $p(x)$ is given by

(24)
$$p^{(k)}(x)=C_k\left(x^{n-k}+G_{n-k}(x)\right)+(n-1)(n-2)\cdots(n-k)\sum_{l=k+1}^{n-1}\frac{G_{l-k}(x)\,x^{n-l}}{(n-l)(l-k)},$$

where

$$C_k=\frac{\sum_{j=1}^{k}(n-1)\ldots(n-j+1)(n-j-1)\ldots(n-k)}{n-k}\ (k=1,2,\ldots,n-1),\ C_0=0.$$

We want to note that

$$p^{(n)}(x)=\left(p^{(n-1)}(x)\right)'=C_{n-1}(x+G_1(x))=C_{n-1}=(n-1)!H_{n-1},$$

where H_{n-1} are called Harmonic numbers, which are defined by

$$H_{n-1}=\sum_{j=1}^{n-1}\frac{1}{j}.$$

With the properties of Genocchi basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, $p(x)$ is introduced by

$$p(x)=\sum_{k=0}^{n}a_kG_k(x).\tag{25}$$

By expression of (25), we obtain that

$$\begin{aligned}a_k&=\frac{1}{2k!}\left(p^{(k-1)}(1)+p^{(k-1)}(0)\right)\\&=\frac{C_{k-1}}{2k!}\left(1+2\delta_{1,n-k+1}\right)+\frac{(n-1)!}{2k!\,(n-k)!}\sum_{l=k}^{n-1}\frac{\left(G_{l-k+1}(1)+0^{n-l}G_{l-k+1}\right)}{(n-l)(l-k+1)}\\&=\frac{C_{k-1}}{2k!}-\frac{\binom{n}{k}}{2n}\sum_{l=k}^{n-1}\frac{(2-G_{l-k+1})}{(n-l)(l-k+1)}.\end{aligned}$$

As a result,

$$a_n=\frac{1}{2n!}\left(p^{(n)}(1)+p^{(n)}(0)\right)=\frac{C_{n-1}}{n!}=\frac{H_{n-1}}{n}.$$

In [8], it is well-known that

$$\frac{C_{k-1}}{k!}=\frac{\binom{n}{k}}{(n-k+1)}\left(H_{n-1}-H_{n-k}\right).\tag{26}$$

By (24), (25) and (26), then we can state the following theorem.

Theorem 2.6. *The following equality*

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)x^{n-k}$$

$$=\sum_{k=0}^{n}\left(\frac{\binom{n}{k}}{2(n-k+1)}(H_{n-1}-H_{n-k})-\frac{\binom{n}{k}}{2n}\sum_{l=k}^{n-1}\frac{(2-G_{l-k+1})}{(n-l)(l-k+1)}\right)G_k(x)$$

holds true.

3. **Further Remarks**

Let $\mathcal{P}_n=\left\{\sum_{j=0}a_jx^j \mid a_j\in\mathbb{Q}\right\}$ be the space of polynomials of degree less than or equal to n. In this final section, we will give the matrix formulation of Genocchi polynomials. Let us now consider the polynomial $p(x)\in\mathcal{P}_n$ as a linear combination of Genocchi basis polynomials with

$$p(x)=C_0G_0(x)+C_1G_1(x)+\cdots+C_nG_n(x).$$

We can write the above as a product of two variables

$$p(x)=\begin{pmatrix} G_0(x) & G_1(x) & \cdots & G_n(x)\end{pmatrix}\begin{pmatrix} C_0\\ C_1\\ \vdots\\ C_n\end{pmatrix}. \tag{27}$$

From expression of (27), we consider the following equation:

$$p(x)=\begin{pmatrix} 1 & x & x^2 & \cdots & x^n\end{pmatrix}\begin{pmatrix} 0 & g_{12} & g_{13} & \cdots & g_{1n+1}\\ 0 & 0 & g_{23} & \cdots & g_{2n+1}\\ 0 & 0 & 0 & \cdots & g_{3n+1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} C_0\\ C_1\\ C_2\\ \vdots\\ C_n\end{pmatrix}$$

where g_{ij} are the coefficients of the power basis that are used to determine the respective Genocchi polynomials. We now list a few Genocchi polynomials as follows:

$$G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\ G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2-1,\cdots.$$

In the quadratic case ($n=2$), the matrix representation is

$$p(x)=\begin{pmatrix} 1 & x & x^2\end{pmatrix}\begin{pmatrix} 0 & 1 & -1\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} C_0\\ C_1\\ C_2\end{pmatrix}.$$

In the cubic case ($n=3$), the matrix representation is

$$p(x)=\begin{pmatrix} 1 & x & x^2 & x^3\end{pmatrix}\begin{pmatrix} 0 & 1 & -1 & 0\\ 0 & 0 & 2 & -3\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Throughout this paper, many considerations for Genocchi polynomials seem to be useful for a matrix formulation.

References

[1] H. Jolany and H. Sharifi, Some results for the Apostol-Genocchi polynomials of higher order, Bulletin of Malaysian Mathematical Sciences Society (in press).

[2] H. Jolany, R. E. Alikelaye and S. S. Mohamad, Some results on the generalization of Bernoulli, Euler and Genocchi polynomials, Acta Universitatis Apulensis, No. 27, 2011, pp. 299-306.

[3] Yuan He and Chunping Wang, Some Formulae of Products of the Apostol-Bernoulli and Apostol-Euler Polynomials, Discrete Dynamics in Nature and Society, vol. 2012, Article ID 927953, 11 pages, 2012. doi:10.1155/2012/927953.

[4] T. Kim, Identities involving Frobenius–Euler polynomials arising from non-linear differential equations, *Journal of Number Theory*, Volume 132, Issue 12, December 2012, Pages 2854-2865.

[5] D. S. Kim, T. Kim, S. H. Lee, Y. H. Kim, Some identities for the product of two Bernoulli and Euler polynomials, *Advances in Difference Equations* 2012 2012:95.

[6] T. Kim and D. S. Kim, Extended Laguerre polynomials associated with Hermite, Bernoulli and Euler numbers and polynomials, *Abstract and Applied Analysis*, Volume 2012, Article ID 95730, 13 pages (Article in press).

[7] T. Kim, D. S. Kim and D. V. Dolgy, Some identities on Bernoulli and Hermite polynomials associated with Jacobi polynomials, *Discrete Dynamics in Nature and Society*, Volume 2012, Article ID 584643, 10 pages (Article in press).

[8] D. S. Kim, D. V. Dolgy, T. Kim, and S. H. Rim, Some formulae for the product of two Bernoulli and Euler polynomials, *Abstract and Applied Analysis*, Volume 2012, Article ID 784307, 15 pages.

[9] D. S. Kim and T. Kim, Euler basis, identities and their applications, *International Journal of Mathematics and Mathematical Sciences*, Volume 2012, Article ID 343981, 13 pages (Article in press).

[10] D. S. Kim and T. Kim, Bernoulli basis and the product of several Bernoulli polynomials, *International Journal of Mathematics and Mathematical Sciences*, Volume 2012, Article ID 463659, 13 pages (Article in press).

[11] T. Kim, Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, *Adv. Stud. Contemp. Math.* 20, 23–28 (2010).

[12] T. Kim, Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.* 16 (2009), No. 4, 484-491.

[13] T. Kim, On the q-extension of Euler and Genocchi numbers, *J. Math. Anal. Appl.* 326 (2007) 1458-1465.

[14] M. Acikgoz and Y. Simsek, On multiple interpolation functions of the Nörlund-type q-Euler polynomials, *Abstract and Applied Analysis*, Volume 2009, Article ID 382574, 14 pages.

[15] E. Cetin, M. Acikgoz, I. N. Cangul and S. Araci, A note on the (h, q)-Zeta type function with weight α, http://arxiv.org/pdf/1206.5299.pdf.

[16] I. N. Cangul, H. Ozden, and Y. Simsek, Generating functions of the (h, q) extension of twisted Euler polynomials and numbers, *Acta Mathematica Hungarica*, vol. 120, no. 3, pp. 281–299, 2008.

[17] S. Araci, M. Acikgoz and F. Qi, On the q-Genocchi numbers and polynomials with weight zero and their interpolation function, http://arxiv.org/pdf/1202.2643v1.pdf.

[18] Q. M. Luo, B. N. Guo, F. Qi, and L. Debnath, Generalization of Bernoulli numbers and polynomials, *IJMMS*, Vol. 2003, Issue 59, 2003, 3769-3776.

[19] S. Araci, D. Erdal and J. J. Seo, A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials, *Abstract and Applied Analysis*, Volume 2011, Article ID 649248, 10 pages.

[20] S. Araci, J. J. Seo, D. Erdal, New construction weighted (h, q)-Genocchi numbers and polynomials related to zeta type functions, *Discrete Dyn. Nat. Soc.* 2011, Art. ID 487490, 7 pp.

[21] A. Bayad, T. Kim, Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials, *Russ. J. Math. Phys.* 18 (2011), no. 2, 133-143.

[22] T. Kim, Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.* 16 (2009), no. 1, 93-96.
[23] N.S. Jung, H. Y. Lee, C. S. Ryoo, Some relations between twisted (h, q)-Euler numbers with weight α and q-Bernstein polynomials with weight α, *Discrete Dyn. Nat. Soc.* 2011, Art. ID 176296, 11 pp.
[24] S. Araci, D. Erdal, and D. J. Kang, Some new properties on the q-Genocchi numbers and polynomials associated with q-Bernstein polynomials, *Honam Mathematical J.* 33 (2011) no. 2, pp. 261-270.
[25] S. Araci, M. Acikgoz, K. H. Park and H. Jolany, On the unification of two families of multiple twisted type polynomials by using p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$, *Bulletin of the Malaysian Mathematical Sciences and Society* (accepted for publication).
[26] S. Araci, M. Acikgoz and K. H. Park, A note on the q-analogue of Kim's p-adic *log* gamma type functions associated with q-extensions of Genocchi and Euler numbers with weight α, *Bulletin of the Korean Mathematical Society* (accepted for publication).

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: `mtsrkn@hotmail.com, saraci88@yahoo.com.tr`

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: `acikgoz@gantep.edu.tr`

Department of Mathematics, Faculty of Science and Letters, Namık Kemal University, 59030 Tekirdağ, TURKEY
E-mail address: `erdogan.math@gmail.com`